MAC 2311

Chapter 3 Exam (3.7 – 3.11) + derivatives
Spring 2020

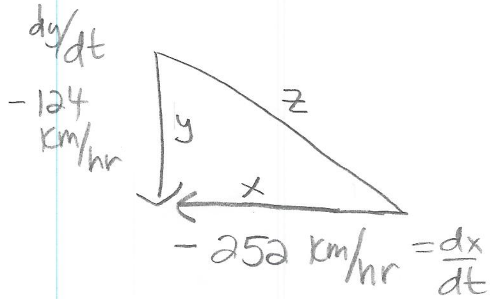
Name KEY
Date: ____________________

This Exam is worth 100 points total. You MUST show your work to receive any credit. Each problem is worth 4 points unless otherwise specified.

Solve the problem. Round your answer, if appropriate.

1) One airplane is approaching an airport from the north at 124 km/hr. A second airplane approaches from the east at 252 km/hr. Find the rate at which the distance between the planes changes when the southbound plane is 34 km away from the airport and the westbound plane is 16 km from the airport. (10 points) 1) C

A) -109 km/hr B) -328 km/hr C) -219 km/hr D) -438 km/hr



$x^2+y^2=z^2$

$2x\frac{dx}{dt}+2y\frac{dy}{dt}=2z\frac{dz}{dt}$

$x^2+y^2=z^2$

$x = 16$ km

$y = 34$ km

$z =$

$16^2+34^2=z^2$

$1412 = z^2$

$38 = z$

$x\frac{dx}{dt}+y\frac{dy}{dt}=z\frac{dz}{dt}$

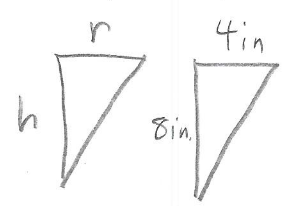
$(16)(-252)+(34)(-124)$

$=(38)\frac{dz}{dt}$

$-217 \text{ km/hr} = \frac{dz}{dt}$

2) Water is being drained from a container which has the shape of an inverted right circular cone. The container has a radius of 4.00 inches at the top and a height of 8.00 inches. At the instant when the water in the container is 6.00 inches deep, the surface level is falling at a rate of 0.5 in./sec. Find the rate at which water is being drained from the container. (10 points) 2) C

A) 23.6 in.^{3}/s B) 13.5 in.^{3}/s C) 14.1 in.^{3}/s D) 20.4 in.^{3}s



$\frac{h}{8}=\frac{r}{4}$

$r=\frac{1}{2}h$

$V=\frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h$

$V=\frac{1}{3}\pi\left(\frac{1}{4}h^2\right)(h)$

$V=\frac{1}{12}\pi h^3$

$\frac{dV}{dt}=\frac{1}{4}\pi h^2\frac{dh}{dt}$

$V=\frac{1}{3}\pi r^2 h$

$\frac{dV}{dt}=\frac{1}{4}\pi(6)^2(-0.5)$

$\boxed{\frac{dV}{dt}=-14.1 \text{ in}^3/\text{s}}$

Provide an appropriate response.

3) Verify the identity using the definitions of hyperbolic functions. 3) ____________

$$\coth x = \frac{e^{2x}+1}{e^{2x}-1}$$

$$\boxed{\coth x = \frac{\cosh x}{\sinh x} = \frac{e^{x}+e^{-x}}{e^{x}-e^{-x}} \cdot \frac{e^{x}}{e^{x}} = \frac{e^{2x}+1}{e^{2x}-1}}$$

Compute dy/dx for the function.

4) y = ln sinh 2x 4) ____________

$$\boxed{2\coth(2x)}$$

$$y' = \frac{1}{\sinh 2x} \cdot \frac{d}{dx}(\sinh 2x)$$

$$y' = \frac{1}{\sinh 2x}(\cosh 2x) \cdot 2$$

$$y' = \frac{2\cosh(2x)}{\sinh(2x)} = 2\coth(2x)$$

Evaluate the derivative.

5) $f(x) = \operatorname{csch}^{-1}(5/x)$ 5) ____________

$$\frac{1}{\sqrt{x^2+25}}$$

(*) the formula should be (+). It faded when copied.

$$\operatorname{csch}^{-1}\left(\frac{5}{x}\right) = \frac{1}{\sin^{-1}\left(\frac{5}{x}\right)}$$

chain $\frac{d}{dx}(5x^{-1}) = -5x^{-2}$

$$\sinh^{-1}\left(\frac{5}{x}\right) = \ln\left(\frac{5}{x} + \sqrt{\left(\frac{5}{x}\right)^2+1}\right)$$

also $\frac{d}{dx}\left(\operatorname{csch}^{-1}(x)\right) = \frac{-1}{|x|\sqrt{1+x^2}}$

let $\frac{5}{x} = x$

$$\frac{d}{dx}\left(\operatorname{csch}^{-1}\left(\frac{5}{x}\right)\right) = \left(\frac{-1}{\left(\frac{5}{x}\right)\sqrt{1+\left(\frac{5}{x}\right)^2}}\right)\left(\frac{-5}{x^2}\right) = \left(\frac{-1}{\left(\frac{5}{x}\right)\sqrt{\frac{x^2+25}{x^2}}}\right)\left(\frac{-5}{x^2}\right)$$

$$= \left(\frac{-1}{\frac{5}{x}\frac{\sqrt{x^2+25}}{x}}\right)\left(\frac{-5}{x^2}\right) = \boxed{\frac{1}{\sqrt{x^2+25}}}$$

Evaluate the expression without using a calculator, or state that the value does not exist. Simplify the answer to the extent possible.

6) sinh(2 ln 7)

6) $\frac{1200}{49}$ 24.5

$\sinh(x) = \frac{e^{x} - e^{-x}}{2}$

$\sinh(2\ln 7) = \sinh(\ln 7^2)$
$= \sinh(\ln 49) = \frac{e^{\ln 49} - e^{-\ln 49}}{2}$

$= \frac{49 - \frac{1}{49}}{2} = \frac{1}{2}\left(\frac{49^2 - 1}{49}\right) = \frac{1200}{49}$

Solve the problem.

7) Suppose that the amount of oil pumped from a well decreases at the continuous rate of 9% per year. When, to the nearest year, will the well's output fall to one-eighth of its present value?

A) 15 years B) 23 years C) 35 years D) 2 years

7) B

$A = A_0 e^{kt}$

$A = A_0 e^{-0.09t}$

$\frac{1}{8}A = Ae^{-0.09t}$

$\frac{1}{8} = e^{-0.09t}$

$\ln\left(\frac{1}{8}\right) = \ln e^{-0.09t}$

$\ln\left(\frac{1}{8}\right) = -0.09t(\ln e)$

$\frac{\ln\left(\frac{1}{8}\right)}{(-0.09)(\ln e)} = t = 23.1 \text{ yrs}$

The function s = f(t) gives the position of a body moving on a coordinate line, with s in meters and t in seconds.

8) $s = 7t^2 + 3t + 8, 0 \le t \le 2$

Find the body's displacement and average velocity for the given time interval.

A) 34 m, 17 m/sec B) 50 m, 25 m/sec C) 20 m, 31 m/sec D) 34 m, 34 m/sec

8) A

$s(t) = 7t^2 + 3t + 8$

$v(t) = 14t + 3$

$s(2) = 7(2)^2 + 3(2) + 8 = 42$

$s(0) = 7(0)^2 + 3(0) + 8 = 8$

$\boxed{\text{average velocity}} = \frac{s(2) - s(0)}{2 - 0}$

$= \frac{42 - 8}{2 - 0} = \frac{34}{2} = 17 \text{ m/s}$

displacement

$7t^2 + 3t + 8 = 0$

$t = \frac{-3 \pm \sqrt{9 - 4(7)(8)}}{2(7)}$

$t =$ not real

$\Rightarrow$ all forward motion

displacement

$= 42 - 8 = 34 \text{ m}$

Solve the problem.

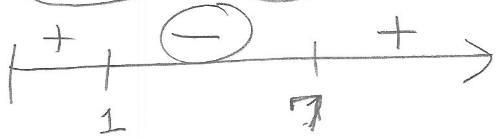
9) At time $t \geq 0$, the velocity of a body moving along the s–axis is $v = t^2 - 8t + 7$. When is the body moving backward? 9) A

A) $1 < t < 7$ B) $0 \leq t < 1$ C) $0 \leq t < 7$ D) $t > 7$

$t^2 - 8t + 7 = 0$

$(t - 7)(t - 1) = 0$

$t = 7$ $t = 1$



test points

$t = \frac{1}{2}$ $(0.5)^2 - 8(0.5) + 7 = +$

$t = 2$ $(2)^2 - 8(2) + 7 = -$

$t = 8$ $(8)^2 - 8(8) + 7 = +$

Find the derivative.

10) $s = t^3 \cos t - 10t \sin t - 10 \cos t$ 10) B

A) $\frac{ds}{dt} = t^3 \sin t - 3t^2 \cos t + 10t \cos t$

B) $\frac{ds}{dt} = -t^3 \sin t + 3t^2 \cos t - 10t \cos t$

C) $\frac{ds}{dt} = -3t^2 \sin t - 10 \cos t + 10 \sin t$

D) $\frac{ds}{dt} = -t^3 \sin t + 3t^2 \cos t - 10t \cos t - 20 \sin t$

$t^3 \cos t$ product

$f = t^3$ $g = \cos t$

$f' = 3t^2$ $g' = -\sin t$

$3t^2 \cos t - t^3 \sin t$

product

$-10t \cdot \sin t$

$f = -10t$ $g = +\sin t$

$f' = -10$ $g' = \cos t$

$-10 \sin t - 10t \cos t$

$s' = 3t^2 \cos t - t^3 \sin t \cancel{-10 \sin t} - 10t \cos t \cancel{+10 \sin t}$

Find the derivative of the function.

11) $f(t) = (6 - t)(6 + t^3)^{-1}$

11) ______________

$f = 6 - t \quad g = (6+t^3)^{-1}$

$f' = -1 \quad g' = -1(6+t^3)^{-2}(3t^2)$

$= -3t^2(6+t^3)^{-2}$

$= -1(6+t^3)^{-1} - 3t^2(6+t^3)^{-2}(6-t)$

$= (6+t^3)^{-2}\left[-1(6+t^3) - 3t^2(6-t)\right]$

$= (6+t^3)^{-2}\left[-6 - t^3 - 18t^2 + 3t^3\right]$

$= \boxed{\dfrac{2t^3 - 18t^2 - 6}{(6+t^3)^2}}$

Find dy/dt.

12) $y = \cos^6(\pi t - 9)$

12) ______________

$y = (\cos(\pi t - 9))^6$

$\boxed{y' = 6(\cos(\pi t - 9))^5 \cdot (-\sin(\pi t - 9))(\pi)}$

13) $y = \cos(\sqrt{8t + 11})$

13) ______________

$y' = -\sin(\sqrt{8t+11}) \cdot \frac{d}{dt}(8t+11)^{1/2}$

$y' = -\sin(\sqrt{8t+11}) \cdot \frac{1}{2}(8t+11)^{-1/2}(8)$

$\boxed{y' = \dfrac{-4\sin(\sqrt{8t+11})}{\sqrt{8t+11}}}$

Use implicit differentiation to find dy/dx.

14) $\cos xy + x^6 = y^6$ 14) D

A) $\dfrac{6x^5 - x\sin xy}{6y^5}$ B) $\dfrac{6x^5 + y\sin xy}{6y^5 - x\sin xy}$ C) $\dfrac{6x^5 + x\sin xy}{6y^5}$ D) $\dfrac{6x^5 - y\sin xy}{6y^5 + x\sin xy}$

$$-\sin(xy)\cdot\left(x\cdot\frac{dy}{dx} + y\right) + 6x^5 = 6y^5\frac{dy}{dx}$$

$$-\sin(xy)\cdot x\cdot\frac{dy}{dx} - \sin(xy)\cdot y + 6x^5 = 6y^5\frac{dy}{dx}$$

$$6y^5 + x\sin(xy)\boxed{\frac{dy}{dx}} = 6x^5 - y\sin(xy) \quad \boxed{\frac{dy}{dx} = \frac{6x^5 - y\sin(xy)}{6y^5 + x\sin(xy)}}$$

Find the derivative of y with respect to x, t, or θ, as appropriate.

15) $y = \ln(\cos(\ln\theta))$ 15) ________

$$y' = \frac{1}{\cos(\ln\theta)}\cdot\underbrace{\frac{d}{d\theta}(\cos(\ln\theta))}_{-\sin(\ln\theta)\cdot\underbrace{\frac{d}{d\theta}(\ln(\theta))}_{\frac{1}{\theta}}} = \boxed{\frac{-\sin(\ln\theta)}{\theta\cos(\ln\theta)}} = \boxed{-\frac{\tan(\ln\theta)}{\theta}}$$

16) $y = \ln\dfrac{1 - x}{(x + 5)^5}$ 16) ________

$$y = \ln(1-x) - \ln(x+5)^5$$

$$y = \ln(1-x) - 5\ln(x+5)$$

$$y' = \frac{-1}{1-x} - 5\left(\frac{1}{x+5}\right) = \boxed{\frac{-1}{1-x} - \frac{5}{x+5}} = \boxed{\frac{4x-10}{(x+5)(1-x)}}$$

Find the derivative.

17) $f(x) = \sin 5e^{-2x}$ 17) ________

$$f'(x) = \cos(5e^{-2x})\cdot\underbrace{\frac{d}{dx}(5e^{-2x})}_{5e^{-2x}(-2)}$$

$$\boxed{f'(x) = -10e^{-2x}\cos(5e^{-2x})}$$

$\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}}$

Find the derivative of y with respect to x.

18) $y = \sin^{-1}\left(\frac{14x + 13}{11}\right)$ 18) B

A) $-\frac{14}{\sqrt{121 - (14x + 13)^2}}$

B) $\frac{14}{\sqrt{121 - (14x + 13)^2}}$

C) $\frac{154}{\sqrt{1 + (14x + 13)^2}}$

D) $\frac{14}{1 + (14x + 13)^2}$

$$y' = \frac{1}{\sqrt{1 - \left(\frac{14x+13}{11}\right)^2}} \cdot \frac{14}{11} = \frac{14}{11} \cdot \frac{1}{\sqrt{\frac{121 - (14x+13)^2}{121}}}$$

$$= \frac{14}{11} \cdot \frac{11}{\sqrt{121 - (14x+13)^2}}$$

Use logarithmic differentiation to find the derivative of y.

19) $y = \frac{x\sqrt{x^3 + 4}}{(x - 1)^{1/3}}$ 19) D

A) $\frac{1}{x} + \frac{3x^2}{2x^3 + 8} - \frac{1}{3x - 3}$

B) $\ln x + \frac{1}{2}\ln(x^3 + 4) - \frac{1}{3}\ln(x - 1)$

C) $\frac{x\sqrt{x^3 + 4}}{(x - 1)^{1/3}}\left[\ln x + \frac{1}{2}\ln(x^3 + 4) - \frac{1}{3}\ln(x - 1)\right]$

D) $\frac{x\sqrt{x^3 + 4}}{(x - 1)^{1/3}}\left[\frac{1}{x} + \frac{3x^2}{2x^3 + 8} - \frac{1}{3x - 3}\right]$

$$\ln y = \ln \frac{x\sqrt{x^3+4}}{(x-1)^{1/3}}$$

$$\ln y = \ln x + \ln (x^3+4)^{1/2} - \ln (x-1)^{1/3}$$

$$\ln y = \ln x + \frac{1}{2}\ln(x^3+4) - \frac{1}{3}\ln(x-1)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{x} + \frac{1}{2}\cdot\frac{1}{x^3+4}\cdot 3x^2 - \frac{1}{3}\cdot\frac{1}{x-1}$$

$$\boxed{\begin{aligned}\frac{dy}{dx} &= \left(\frac{1}{x} + \frac{3x^2}{2(x^3+4)} - \frac{1}{3(x-1)}\right) y \\ \frac{dy}{dx} &= \left(\frac{1}{x} + \frac{3x^2}{2(x^3+4)} - \frac{1}{3(x-1)}\right)\left(\frac{x\sqrt{x^3+4}}{(x-1)^{1/3}}\right)\end{aligned}}$$